

Physical and Geometric Interpretations of the Riemann Tensor, Ricci Tensor, and Scalar Curvature

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Abstract

Various interpretations of the Riemann Curvature Tensor, Ricci Tensor, and Scalar Curvature are described. Also the physical meanings of the Einstein Tensor and Einstein's Equations are discussed. Finally, a derivation of Newtonian Gravity from Einstein's Equations is given.

1 Introduction

Electromagnetism is taught at a variety of levels with various combinations of intuitive pictures and hard equations. One generally begins (often in elementary school) with such simple concepts as opposites attract, rubbing a balloon on your hair makes it charged, and magnets have two distinct poles. Later Coulomb's Law may be introduced and then the integral forms of Maxwell's equations which have very intuitive pictorial representations. Charges are the sources (and sinks) for all electric charge (Gauss' Law of Electricity) Currents cause circulation of magnetic fields (Ampere's Law). Later the derivative forms and covariant forms of Maxwell's equations can be added building upon these foundations.

Unfortunately, such is generally not the case in teaching General Relativity. Either one is presented with a very vague elementary presentation

about space being like a globe where innitially parallel lines end up crossing, or one gets the full graduate treatment, based on complicated indeces or advanced differential geometry. There is very little middle ground, and very little intuitive explanation at the advanced level.

I believe that part of the reason for this is a lack of well known intuitive geometrical explanations of the important objects in GR such as the Ricci Tensor, Scalar Curvature, and Einstein's Tensor. Such pictures would make it easier to visualize the equations of GR and allow teaching the concepts at a middle level.

Parallel transport is actually fairly well described in most elementary discussions so I will not focus on it. The Riemann Tensor is also often well described in a geometrical sense in more advanced discussions. One could just say that the Ricci Tensor, Scalar Curvature, and Einstein's Tensor are just specific linear combinations of the Riemann Tensor. One could also say that divergences and curls are just linear combinations of partial derivatives on vector fields. That is true, but overlooks the vital facts, that divergence represents the overall increase in flux density, and curl represents the amount of circulation. Both of which are vitally important ideas for a conceptual understanding.

Rather than trying to make a self contained mid-level treatment of GR, this article is written for those like myself who have graduate training in the field, but seek to better understand the fundamental objects so they can better explain them to others. However, the main body of the text should be understandable to any with an intuitive understanding of parallel transport and geodesics.

2 Conventions

I will begin by reviewing my conventions, which in general follow reference [1].

Throughout this paper I will restrict myself to torsion free spaces with a metric and a metric-connection,

$$\Gamma^\kappa_{\mu\nu} = \frac{1}{2} g^{\kappa\lambda} (\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}).$$

The Riemann curvature tensor is then

$$R^\kappa_{\lambda\mu\nu} = \partial_\mu \Gamma^\kappa_{\nu\lambda} - \partial_\nu \Gamma^\kappa_{\mu\lambda} + \Gamma^\eta_{\nu\lambda} \Gamma^\kappa_{\mu\eta} - \Gamma^\eta_{\mu\lambda} \Gamma^\kappa_{\nu\eta}.$$

Vectors will be represented with bold faced roman letters. Vector components will be represented with superscripts. I will also use the following symbols:

$$\begin{aligned} (\mathbf{T}, \mathbf{S}) &= g_{\mu\nu} T^\mu T^\nu \\ R(\mathbf{P}, \mathbf{Q}, \mathbf{S}, \mathbf{T}) &= R_{\mu\nu\rho\sigma} P^\mu Q^\nu S^\rho T^\sigma. \end{aligned}$$

The Ricci tensor is the contraction of the Riemann tensor, and will be written as R with just two indeces or two arguments

$$R_{\mu\nu} = R^\rho_{\mu\rho\nu}.$$

The scalar curvature is the contraction of the Ricci tensor, and is written as R without subscripts or arguments

$$R = g^{\mu\nu} R_{\mu\nu}.$$

Often times, partial derivatives will be represented with a comma

$$\partial_\mu A = A_{,\mu}.$$

3 Gaussian Curvature of a Two-Dimensional Surface

I will begin by describing Gauss' notion of internal curvature. I will basically follow reference [2]. Prior to Gauss people had studied the notion of extrinsic curvature. That is, they studied the way a curve imbedded in a two dimensional space turned as one moved along it, and at any given point they could find the effective radius ρ of a circle that would turn at the same rate as the curve at that point. The curvature was then simply the inverse of this effective radius. $\kappa = 1/\rho$. A two dimensional surface imbedded in a three dimensional space could be described by two such radii or curvatures. In 1827, Gauss had an idea for defining curvature in an intrinsic way, that is in a way that could be measured without embedding the surface in a higher dimensional space. Imagine starting at a point in a space and moving a geodesic

distance ϵ in all directions. In essence you would form a the equivalent of a “circle” in this space. If the space is flat, then the circumference C of this circle will simply be $C = 2\pi\epsilon$ but in a curved space the circumference will be slightly greater or smaller than this. For example on a sphere of radius ρ , the circumference would be

$$C = 2\pi\rho \sin(\epsilon/\rho) \approx 2\pi\epsilon(1 - \frac{\epsilon^2}{6\rho^2})$$

For more complicated surfaces the expression is not so simple, but in general the lowest order deviation will be quadratic as in this case. Gauss, then suggested the following definition for curvature κ_{Gauss} .

$$\kappa_{Gauss} = \lim_{\epsilon \rightarrow 0} \frac{6}{\epsilon^2} \left(1 - \frac{C}{2\pi\epsilon}\right). \quad (1)$$

It turns out that this intrinsic curvature is just the product of the two extrinsic curvatures of the surface.

What is most important for this paper however, is that we have an intrinsic measure of curvature with a clear and concrete physical/geometrical meaning. The Gaussian Curvature κ_{Gauss} is a measure of how much the circumference of a small circle deviates from its expected value in a flat space. The deviation is quadratic, as we would expect since our usual measure of curvature, the Riemann tensor, depends on second derivatives of the metric. We will find that all of our physical interpretations of curvature will depend on quadratic variations. Constant and linear variations are simply artifacts of the coordinate system.

There are a number of other relationships between the gaussian curvature and physical quantities. However, they are well detailed in the literature [2] and this one interpretation is sufficient for this paper.

4 Interpretations of the Rieman Tensor

There are a variety of physical/geometrical interpretations of the Riemann Tensor. Most texts on gravitation or Riemannian geometry will present at least one of them. Therefore I will present them only briefly.

4.1 Collection of Gaussian Curvatures

I believe this interpretation is popular among mathematicians, and can be found in [2]. If you have two non-parallel vectors S and T at a point in your space, then they define a two dimensional subspace. The quantity

$$R_{\mu\nu\rho\sigma} S^\mu T^\nu S^\rho T^\sigma$$

is simply the gaussian curvature of that subspace times the area squared of the \mathbf{S} \mathbf{T} parallelogram. In other words

$$\kappa_{G,T \wedge S} \begin{vmatrix} (\mathbf{S}, \mathbf{S}) & (\mathbf{S}, \mathbf{T}) \\ (\mathbf{T}, \mathbf{S}) & (\mathbf{T}, \mathbf{T}) \end{vmatrix} = R(\mathbf{S}, \mathbf{T}, \mathbf{S}, \mathbf{T}). \quad (2)$$

This equation can be proven by direct calculation of the Gaussian Curvature as described above. (Define a “circle”, find its circumference, and find its deviation from the circumference in flat space. This method is followed in depth for D dimensions in appendix B. Here only two dimensions are required.) The calculation is straight forward though tedious, and care must be taken to include all second order variations, including changes in the metric. The area dependence is most easily found by first allowing \mathbf{S} and \mathbf{T} to be orthonormal, and then generalizing.

4.2 Deviation of a Vector

This is perhaps the most common interpretation used in physics texts and can be found in [1], [2], or [3]. Begin with a vector A and parallel transport it around a small parallelogram defined by the vectors \mathbf{u} and \mathbf{v} (starting with displacement along \mathbf{u}). The deviation of the vector \mathbf{A} can then be found as

$$\delta A^\alpha = -R^\alpha_{\beta\mu\nu} A^\beta u^\mu v^\nu. \quad (3)$$

This again can be shown by straight forward calculation carrying all terms to second order.

4.3 Geodesic Deviation Equation

This is another common interpretation used in physics texts, including [3] that is very useful for understanding General Relativity. Consider a one

parameter family of geodesics $x^\mu(s, t)$ where s is the parameter and t the geodesic flow. $S^\mu = dx^\mu/ds$ is the vector describing motion from one geodesic to another and $T^\mu = dx^\mu/dt$ describes flow along the geodesic. Then we find that if \mathbf{S} is parallel transported along the geodesic flow, its second covariant derivative is determined by the Riemann tensor:

$$\frac{D^2 S^\mu}{dt^2} = -R^\mu_{\nu\sigma\rho} S^\sigma T^\nu T^\rho. \quad (4)$$

While the description of this equation may seem forboding, its meaning can be made more physically clear (if somewhat less technically accurate) by imagining that \mathbf{S} represents the separation between two objects near each other in space and \mathbf{T} represents their initial motion. The equation then simply describes their relative acceleration.

5 Interpretations of the Ricci Tensor

We now turn our attention to the Ricci Tensor. The ricci tensor is simply a contraction of the Rieman tensor, defined as

$$R_{\mu\nu} = R^\rho_{\mu\rho\nu}. \quad (5)$$

Thus, if we know all of the Riemann tensor we can calculate the Ricci tensor, but we haven't really addressed the issue of its meaning. (In the same sense that saying $\nabla \mathbf{E}$ is just a linear combination of $\partial_i E_j$ does not explain that the gradient must equal the sources of the field.)

5.1 Sum of Gaussian Curvatures

Whenever I've asked a mathematician what the Ricci tensor means, they've explained the meaning of the Riemann tensor as a collection of gaussian curvatures and simply stated that the Ricci tensor was an average. While I do not find this explanation very satisfying, it bears further investigation. Suppose you wanted to find the average curvature of all planes involving the vector \mathbf{S} . You could start by taking a collection of orthonormal vectors \mathbf{t}_i and saying

$$\bar{\kappa}_{\mathbf{S}} = \frac{1}{D} \sum_{i=1}^D R(\mathbf{S}, \mathbf{t}_i, \mathbf{S}, \mathbf{t}_i) = \frac{1}{D} R(\mathbf{S}, \mathbf{S}). \quad (6)$$

We should note that these are actually area weighted curvatures. That is they contain a factor of the area squared of the parallelogram formed by \mathbf{S} and \mathbf{t}_i , as in equation 2.

One may ask whether summing over one orthonormal set is sufficient. Perhaps we should average over all such orthonormal sets. The Ricci tensor does not depend on your basis, so no further averaging is required. So while the Riemann tensor told us the gaussian curvature of any given sub plane, the Ricci tensor gives us the average of all sub planes involving a given vector.

5.2 Volume Deviation

I find the previous description lacking because instead of describing the behavior of a single physical object, it describes an average of the behavior of several objects. There is a way to describe the physical meaning of the Ricci tensor without invoking the notion of an average. I found the idea for this description in [4], but proofs and full details were not included.

Suppose instead of looking at two small objects in space, we considered a volume filling collection of small objects in space. Describing the relative acceleration of any two of them would require the geodesic deviation equation, but to describe the evolution of their volume, we would have to average over several different versions of the equation. These have roughly the result of averaging the Riemann tensor into a Ricci tensor. So in roughly the same sense that the Riemann tensor governs the evolution of a vector or a displacement parallel propagated along a geodesic, the Ricci tensor governs the evolution of a small volume parallel propagated along a geodesic. We must be careful though. Unlike vectors, volumes may change along geodesics even in a flat space. We must therefore subtract off any change that would occur in a flat space. Suppose then that we have a small volume δV of dust near a point x_0^μ . If we allow that volume to move along a direction $T^\mu = \frac{dx^\mu}{d\tau}$ we find the following equation:

$$\frac{D^2}{d\tau^2}\delta V - \frac{D_{\text{flat}}^2}{d\tau^2}\delta V = -\delta V R_{\mu\nu} T^\mu T^\nu. \quad (7)$$

The proof of this equation can be found in appendix A.

One may ask whether this volume should be a D -dimensional volume (in relativity a space time volume) or a $D - 1$ -dimensional volume (a space only volume). It turns out that it can be either, so long as the $D - 1$ -dimensional

volume is transverse to the vector \mathbf{T} . Thus, we may apply the equation to the deviation of a spacelike volume as it propagates through time.

6 Interpretation of the Scalar Curvature

We turn now to the scalar curvature, which is just the contraction of the Ricci tensor. It turns out the scalar curvature has a meaning very similar to the gaussian curvature. If we imagine instead of taking a circle, taking a generalized $D - 1$ sphere, i.e. the set of all points a geodesic distance ϵ from a given starting point x_0^μ . We can calculate the area of this sphere in flat space, but in curved space the area will deviate from the one we calculated by an amount proportional to the curvature. Thus, we find that the scalar curvature is

$$R = \lim_{\epsilon \rightarrow 0} \frac{6D}{\epsilon^2} \left[1 - \frac{A_{\text{curved}}(\epsilon)}{A_{\text{flat}}(\epsilon)} \right]. \quad (8)$$

The proof of this equation is in appendix B.

Feynman [5] mentions this definition of curvature in the case of three space dimensions. His, definition differs from the one given here by a factor of -2 which will be explained in the next section.

7 The Einstein Tensor

We now have a physical interpretation for each part of Einstein's equation, and we could begin immediately to discuss the equation's meaning. However, I'd like to first discuss the particular combination of the Ricci tensor and scalar curvature known as Einstein's tensor.

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$$

Feynman [5] discusses the meaning of this tensor to some extent, and I will elaborate on his ideas, though my presentation will be very different.

Suppose we know the Riemann tensor for some point in a D dimensional space, but that we wanted to find the scalar curvature not for the full space, but for the $D - 1$ subspace that is orthogonal to a particular vector \mathbf{t} . In the specific case of General Relativity in four dimensions, \mathbf{t} could represent the direction of time and we would then be finding the curvature of the three

coresponding spacial dimensions. We would have to contract the Riemann tensor, not with $g^{\mu\nu}$ but rather with the projector onto our $D - 1$ space

$$g^{\mu\nu} - t^\mu t^\nu.$$

(Here it is assumed that \mathbf{t} is normalized to 1.) The $D - 1$ dimmensional curvature is then

$$R_{D-1} = (g^{\mu\nu} - t^\mu t^\nu)(g^{\alpha\beta} - t^\alpha t^\beta)R_{\mu\alpha,\nu\beta} = R - 2R_{\mu\nu}t^\mu t^\nu = -2G_{\mu\nu}t^\mu t^\nu. \quad (9)$$

So we see that the curvature of the $D - 1$ dimmensional subspace orthogonal to the unit vector \mathbf{t} is just negative two times the Einstein tensor fully contracted with the vector \mathbf{t} . For general relativity, this means that once we choose a time direction, Einstein's tensor tells us the scalar curvature of the corresponding spacial dimensions.

In Feynman's treatment, he apparently includes the factor of -2 in his deffinition of the scalar curvature. This accounts for the difference in our definition of scalar curvature in three dimensional space. The numerical factor is unimportant so long as it is treated correctly. The sign change compensates for the fact that the spatial dimmensions have a negative metric. Feynman's sign convention gives the curvature that would be found if we treated only the spatial dimensions with a positive metric. I will continue with my sign convention and make comments where necesary.

We can now discuss the meaning of Eintein's Equation which reads

$$G_{\mu\nu} = 8\pi GT_{\mu\nu},$$

where $T_{\mu\nu}$ is the stress-energy tensor, and G is the Universal Gravitational Constant. When the stress-energy tensor is contracted with a timelike unit vector \mathbf{t} we get the energy density ρ in the frame of reference defined by \mathbf{t} . Thus If we contract Einstein's equation with the vector \mathbf{t} giving the time direction, we get

$$R_3 = -4\pi G\rho. \quad (10)$$

Thus, the scalar curvature of the spatial dimensions equal $-4\pi G$ times the energy density in any chosen frame of reference. This is the meaning of the Einstein equation. Curvature equals energy density, and the equality must hold in all frames.

The sign may appear misleading, as it suggests that positive energy would give a negative curvature. However, this is because the spatial dimensions have negative signature so that closed surfaces such as spheres have negative curvature. Because of Einstein's equation we learn that the earth for example does not actually have the shape of a ball in flat space, but of a ball at one point on a very large three sphere. However, the curvature is very small. As Feynmann points out, if we calculated the change in radius due to this curvature we would find that it is approximately 1 fermi for every 4 billion metric tons.

8 Deriving Newtonian Gravity

Now let's try to use our results to derive Newtonian gravity. We will restrict ourselves to 4 dimensions (3 space, 1 time). First we begin with Einstein's equation written in terms of the Ricci tensor and scalar curvature

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu} \quad (11)$$

and contract both sides to get

$$R = -8\pi GT.$$

Substituting back into equation 11 and rearranging we get

$$R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}). \quad (12)$$

Next we fully contract the equation with the timelike unit vector \mathbf{t} which describes the initial motion of the matter and get

$$R_{\mu\nu}t^\mu t^\nu = 8\pi G(\rho - \frac{1}{2}T).$$

If we specialize to a region with matter dominated energy, then T is simply the energy density in the matter's rest frame ρ_0 . If we allow \mathbf{t} to describe that same rest frame (or one moving very slowly relative to it), then both terms on the right hand side contain the same factor of ρ so that we get,

$$R_{\mu\nu}t^\mu t^\nu = 4\pi G\rho.$$

Next we substitute this into equation 7 and get

$$\frac{D^2}{d\tau^2}\delta V - \frac{D_{\text{flat}}^2}{d\tau^2}\delta V = 4\pi G\rho\delta V.$$

We integrate this over space to put it in terms of a finite spherical volume of radius r so that our equation is

$$\frac{D^2}{d\tau^2}V - \frac{D_{\text{flat}}^2}{d\tau^2}V = 4\pi GM_{\text{enclosed}}.$$

Now let's consider the time derivative. As long as the objects experiencing the gravitational force are moving slowly, and the force is small, the covariant derivative is just a regular derivative with respect to time. Thus, we get

$$\ddot{V} = 4\pi r^2\ddot{r} + 8\pi r\dot{r}^2.$$

The second term on the left is the one which would be there even in flat space (no gravity) The first term is then the one which enters into our equation and we get

$$4\pi r^2\ddot{r} = -4\pi GM_{\text{enc}},$$

or

$$\ddot{r} = -\frac{GM_{\text{enc}}}{r^2}. \quad (13)$$

This is of course Newton's equation for universal gravity (for the specific case of slow moving matter). It should be noted that by using the standard formula for the volume of the sphere we were already ignoring higher order corrections.

It is interesting to note that if we contract equation 12 with a lightlike vector \mathbf{t} instead of a timelike one, then the trace term T drops out and we get an acceleration that is twice as big. This may explain why when describing the bending of light, General Relativity gives a result that is twice as big as the one from Newtonian Gravity.

9 Conclusion

By now we have looked in depth at the various important curvature quantities in General Relativity. We have studied their geometric and physical meanings, and we have seen how the lowest order Newtonian Gravitation is a result of Einstein's equation and the curvature of space. I hope this is helpful for your own understanding of General Relativity and in teaching it.

A Proof of Volume Deviation Equation

We begin by defining a small volume described by a set of D vectors $\{\mathbf{t}_i\}$, where D is the dimension of the space. The vectors must be linearly independent, otherwise they would not span a D -volume. The volume can then be written as

$$\delta V = \prod_{i=1}^D t_i^{\mu_i} \sqrt{g} \epsilon_{\mu_1 \mu_2 \dots \mu_D} = \det(t_i^\mu) = \prod_{\mu=1}^D t_{\alpha_\mu}^\mu \sqrt{g} \epsilon^{\alpha_1 \alpha_2 \dots \alpha_D}. \quad (14)$$

The tensor $\sqrt{g} \epsilon_{\mu_1 \mu_2 \dots \mu_D}$ is covariantly invariant, so when we take covariant derivatives of the volume we need only act on the vectors. We can then take covariant derivatives along a curve described by the vector \mathbf{T} and the parameter τ . The first derivative is

$$\frac{D}{d\tau} \delta V = \sum_{\mu=1}^D \dot{t}_{\alpha_\mu}^\mu \prod_{\substack{\nu=1 \\ \nu \neq \mu}}^D t_{\alpha_\nu}^\nu \sqrt{g} \epsilon^{\alpha_1 \alpha_2 \dots \alpha_D},$$

and the second derivative is

$$\frac{D^2}{d\tau^2} \delta V = \sum_{\mu=1}^D \ddot{t}_{\alpha_\mu}^\mu \prod_{\substack{\nu=1 \\ \nu \neq \mu}}^D t_{\alpha_\nu}^\nu \sqrt{g} \epsilon^{\alpha_1 \alpha_2 \dots \alpha_D} + \sum_{\substack{\mu, \nu=1 \\ \nu \neq \mu}}^D \dot{t}_{\alpha_\mu}^\mu \dot{t}_{\alpha_\nu}^\nu \prod_{\substack{\rho=1 \\ \rho \neq \mu, \nu}}^D t_{\alpha_\rho}^\rho \sqrt{g} \epsilon^{\alpha_1 \alpha_2 \dots \alpha_D}. \quad (15)$$

In these equations

$$\dot{t}_{\alpha_\mu}^\mu = \frac{D}{d\tau} t_{\alpha_\mu}^\mu = \nabla_T t_{\alpha_\mu}^\mu,$$

and

$$\ddot{t}_{\alpha_\mu}^\mu = \frac{D^2}{d\tau^2} t_{\alpha_\mu}^\mu = \nabla_T (\nabla_T t_{\alpha_\mu}^\mu) = -R_{\nu\rho\sigma}^\mu t_{\alpha_\mu}^\rho T^\nu T^\sigma$$

The second term in equation 15 would appear even if the space were flat. It simply refers to the expanding or contracting of the volume in flat space. For example in Euclidean three space we can imagine a volume of constant solid angle $d\Omega$ and constant radial width dr moving radially with constant radial velocity. All points in the volume are moving along geodesics, the space is clearly flat, but the volume is increasing quadratically with time, and therefore has a non-zero second derivative. I do wonder if a rescaling of τ couldn't set this to zero, but at this point I'm uncertain. This is the term

referred to as $\frac{D_{\text{flat}}^2}{d\tau^2} \delta V$ in equation 7. It is subtracted off in that equation, and plays no further role here.

Substituting the geodesic deviation equation 4 into the first term in equation 15 we get

$$\sum_{\mu=1}^D \ddot{t}_{\alpha_\mu}^\mu \prod_{\substack{\nu=1 \\ \nu \neq \mu}}^D t_{\alpha_\nu}^\nu \sqrt{g} \epsilon^{\alpha_1 \alpha_2 \dots \alpha_D} = \sum_{\mu=1}^D -R_{\phi\rho\sigma}^\mu t_{\alpha_\mu}^\rho T^\phi T^\sigma \prod_{\substack{\nu=1 \\ \nu \neq \mu}}^D t_{\alpha_\nu}^\nu \sqrt{g} \epsilon^{\alpha_1 \alpha_2 \dots \alpha_D}.$$

From here we can see that if ρ is equal to any of the ν 's then we are taking the determinant of a matrix with a repeated row and the term is therefore 0. We can therefore set $\rho = \mu$ because μ is the only value not already appearing in the product. The sum then factors out and we have

$$\left(\sum_{\mu=1}^D -R_{\phi\mu\sigma}^\mu \right) (T^\phi T^\sigma) \left(\prod_{\nu=1}^D t_{\alpha_\nu}^\nu \sqrt{g} \epsilon^{\alpha_1 \alpha_2 \dots \alpha_D} \right) = -R_{\phi\sigma} T^\phi T^\sigma \delta V. \quad (16)$$

Thus proving the guess for the Ricci tensor.

For General Relativity, we would like to think of the parameter τ as representing time, and \mathbf{T} as the direction of time flow. We would then generally want to know the evolution of a space volume not a full space-time volume. Thus, we want to know if the above equation still applies if we deal with a $D - 1$ volume. The answer is that a $D - 1$ volume is allowed so long as it is transverse to the vector \mathbf{T} . Because our set of vectors $\{\mathbf{t}_i\}$ spans space(-time) it must be linearly dependent with \mathbf{T} . We can therefore drop one of the \mathbf{t}_i 's in favor of \mathbf{T} without changing the volume. However, the geodesic deviation of \mathbf{T} is zero

$$\ddot{T}^\mu = -R_{\phi\rho\sigma}^\mu T^\phi T^\rho T^\sigma = 0$$

because the Riemann tensor is anti-symmetric in the last two indeces. Thus, this term gives no contribution to the deviation of the volume, and the proof is unchanged.

B General Proof of the Scalar Curvature Relation

It was stated in the text that the scalar curvature tells how a surface area of a generalized sphere in a curved space differs from the surface of a sphere in

a flat space. For our purposes a sphere of radius r at a point x_0 is the set of all points that are a geodesic distance r from the given fixed point x_0 . In this section we will prove the relationship between curvature and boundary area by direct construction.

We begin by considering a point x_0^μ , and finding the coordinates of a point $x^\mu(\tau)$ a distance τ away in an arbitrary direction. Next we will consider how much these coordinates change when we change our direction. That is we will find $\frac{dx^\mu}{d\theta_i}$ where θ_i is a set of variables representing the change in direction. Using these changes in position we can find a small surface area element which we can integrate to find the total surface area of the sphere.

Begin by considering the point x_0^μ and choosing a set of orthonormal vectors $\{t_i^\mu\}$ at this point. This is in essence equivalent to finding the Riemann Normal Coordinates at this point. We will assume from now on that we are using Riemann Normal Coordinates around the point x_0^μ . We can now describe any unit vector t^μ at this point as a linear superposition of these basis vectors. If we use the standard polar angles in D -dimensions, then our arbitrary vector can simply be thought of as the radial unit vector. For example, in two dimensions the radial vector is simply

$$t^\mu(\theta) = t_1^\mu \cos \theta + t_2^\mu \sin \theta.$$

Now we want to find the coordinates of the point a small geodesic distance τ away from the point x_0^μ in the direction t^μ . We start with the geodesic equation

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\sigma} \frac{dx^\nu}{d\tau} \frac{dx^\sigma}{d\tau} = 0$$

and solve it iteratively to get $x^\mu(\tau)$ to order τ^3 :

$$x^\mu(\tau) = x_0^\mu + \tau t^\mu - \frac{\tau^3}{6} \tilde{\Gamma}_{\nu\sigma,\alpha}^\mu t^\nu t^\sigma t^\alpha. \quad (17)$$

It is important to note that because we are in Riemann Normal Coordinates, Γ is 0 to 0th order in τ . That is why there is no term quadratic in τ and why the Γ term involves a derivative. It is useful to symmetrise the Γ term in all three of the lower indices. This can be done because it is already multiplying an object that is symmetric in those indeces. Thus we define

$$\tilde{\Gamma}_{\nu\sigma,\alpha}^\mu = \Gamma_{\nu\sigma,\alpha}^\mu + \Gamma_{\alpha\nu,\sigma}^\mu + \Gamma_{\sigma\alpha,\nu}^\mu$$

so that we can write

$$x^\mu(\tau) = x_0^\mu + \tau t^\mu - \frac{\tau^3}{18} \tilde{\Gamma}_{\nu\sigma,\alpha}^\mu t^\nu t^\sigma t^\alpha.$$

These are the coordinates of the point on our sphere in the direction t^μ from the starting point. A small bit of surface area is then simply

$$dA = \sqrt{\det \left\{ g \left(\frac{dx^\mu}{d\theta_i}, \frac{dx^\mu}{d\theta_j} \right) \right\}} d\theta_1 d\theta_2 \dots d\theta_{D-1}. \quad (18)$$

$\frac{dx^\mu}{d\theta_i}$ is easily found to be

$$\frac{dx^\mu}{d\theta_i} = \tau \frac{dt^\mu}{d\theta_i} - \frac{\tau^3}{6} \tilde{\Gamma}_{\nu\sigma,\alpha}^\mu t^\nu t^\sigma \frac{dt^\alpha}{d\theta_i},$$

but we must be very careful in evaluating the inner product. Namely it must be evaluated not at x_0 but at $x(\tau)$. To do this, we simply Taylor expand the metric to order τ^2 and find

$$g_{\mu\nu}(x(\tau)) = g_{\mu\nu}(x_0) + \frac{\tau^2}{2} g_{\mu\nu,\alpha\beta} t^\alpha t^\beta.$$

The inner product can now be evaluated to get

$$g\left(\frac{dx^\mu}{d\theta_i}, \frac{dx^\mu}{d\theta_j}\right) = g_{\mu,\nu} \tau^2 \frac{dt^\mu}{d\theta_i} \frac{dt^\nu}{d\theta_j} + \frac{\tau^4}{2} (g_{\mu\nu,\alpha\beta} - \frac{2}{3} \tilde{\Gamma}_{\mu\alpha\beta,\nu}) t^\alpha t^\beta \left\{ \frac{dt^\mu}{d\theta_i} \frac{dt^\nu}{d\theta_j} \right\}, \quad (19)$$

where

$$\left\{ \frac{dt^\mu}{d\theta_i} \frac{dt^\nu}{d\theta_j} \right\} = \frac{1}{2} \left(\frac{dt^\mu}{d\theta_i} \frac{dt^\nu}{d\theta_j} + \frac{dt^\mu}{d\theta_j} \frac{dt^\nu}{d\theta_i} \right).$$

After some mild algebra and a few rearrangements involving the symmetry of the i and j terms we can show that the term in brackets is proportional to the Riemann tensor evaluated in Riemann Normal Coordinates:

$$g_{\mu\nu,\alpha\beta} - \frac{1}{3} \tilde{\Gamma}_{\mu\beta\alpha,\nu} - \frac{1}{3} \tilde{\Gamma}_{\nu\beta\alpha,\mu} = -\frac{1}{3} (g_{\mu\alpha,\nu\beta} - g_{\mu\nu,\alpha\beta} - g_{\alpha\beta,\mu\nu} + g_{\nu\beta,\mu\alpha}) = -\frac{2}{3} R_{\mu\beta,\nu\alpha}.$$

Equation 19 then simplifies to

$$g\left(\frac{dx^\mu}{d\theta_i}, \frac{dx^\mu}{d\theta_j}\right) = g_{\mu,\nu} \tau^2 \frac{dt^\mu}{d\theta_i} \frac{dt^\nu}{d\theta_j} - \frac{\tau^4}{3} (R_{\mu\beta,\nu\alpha}) t^\alpha t^\beta \left\{ \frac{dt^\mu}{d\theta_i} \frac{dt^\nu}{d\theta_j} \right\}. \quad (20)$$

The determinant in equation 18 is then

$$\det g\left(\frac{dx^\mu}{d\theta_i}, \frac{dx^\mu}{d\theta_j}\right) = \det\left(g_{\mu,\nu}\tau^2 \frac{dt^\mu}{d\theta_i} \frac{dt^\nu}{d\theta_j} - \frac{\tau^4}{3}(R_{\mu\beta,\nu\alpha})t^\alpha t^\beta \left\{ \frac{dt^\mu}{d\theta_i} \frac{dt^\nu}{d\theta_j} \right\}\right). \quad (21)$$

Before simplifying this determinant, we must make a brief digression and introduce some new structure. Notice that $x^\mu = x_0^\mu + \tau t^\mu$ gives the coordinates of a point. An equally good set of coordinates is $\Theta_I = (\tau, \theta_i)$. If the x^μ represent the Riemann Normal Coordinates, the Θ_I represent the equivalent polar coordinates. The metric in this new set of coordinates is then

$$H_{IJ} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \Theta_I} \frac{\partial x^\nu}{\partial \Theta_J}. \quad (22)$$

From this we can easily find the inverse relationship

$$g^{\mu\nu} = H^{IJ} \frac{\partial x^\mu}{\partial \Theta_I} \frac{\partial x^\nu}{\partial \Theta_J}. \quad (23)$$

The various components of this matrix are

$$H_{00} = g_{\mu\nu} t^\mu t^\nu = 1; \quad (24)$$

$$H_{0i} = \tau g_{\mu\nu} t^\mu \frac{\partial t^\nu}{\partial \theta_j} = 0; \quad (25)$$

$$H_{ij} = \tau^2 g_{\mu\nu} \frac{\partial t^\mu}{\partial \theta_i} \frac{\partial t^\nu}{\partial \theta_j} = \tau^2 h_{ij}. \quad (26)$$

The first equation is simply the normalization of t^μ . The second equation being 0 follows directly from this normalization. If the magnitude of \mathbf{t} is to remain unchanged it must be orthogonal to its deviation. In the third equation, we have introduced a new matrix $h_{ij} = g_{\mu\nu} \frac{\partial t^\mu}{\partial \theta_i} \frac{\partial t^\nu}{\partial \theta_j}$.

Because H^{IJ} is block diagonal we can invert the blocks individually.

$$H^{ij} = \frac{1}{\tau^2} h^{ij};$$

$$H^{00} = 1.$$

Substituting this into equation 23 we find that

$$g^{\mu\nu} = h^{ij} \frac{\partial t^\mu}{\partial \theta_i} \frac{\partial t^\nu}{\partial \theta_j} + t^\mu t^\nu. \quad (27)$$

This will be useful for simplifying our curvature equation.

Now we return to equation 21. Notice that the first term is just $\tau^2 h_{ij}$ and the second term is smaller by a factor of τ^2 . Using a standard perturbative expansion for determinants we find that

$$\det g\left(\frac{dx^\mu}{d\theta_i}, \frac{dx^\mu}{d\theta_j}\right) = \tau^{2(D-1)} (\det h_{ij}) \left(1 - \frac{\tau^2}{3} (R_{\mu\beta,\nu\alpha}) t^\alpha t^\beta \left\{ \frac{dt^\mu}{d\theta_i} \frac{dt^\nu}{d\theta_j} \right\} h^{ij}\right),$$

where higher order terms have been dropped. The first two factors are just what we would get in flat space. We can use them to define a flat space area element $dA_{flat} = d\theta_1 \dots d\theta_{D-1} \tau^{D-1} \sqrt{\det h_{ij}}$. The third factor shows that we now have a small curvature dependent perturbation on the flat space result. This perturbation can be simplified using equation 27.

$$(R_{\mu\beta,\nu\alpha}) t^\alpha t^\beta \left\{ \frac{dt^\mu}{d\theta_i} \frac{dt^\nu}{d\theta_j} \right\} = (R_{\mu\beta,\nu\alpha}) t^\alpha t^\beta (g^{\mu\nu} - t^\mu t^\nu) = R_{\alpha\beta} t^\alpha t^\beta$$

The $g^{\mu\nu}$ term causes the contraction, and the $t^\mu t^\nu$ term cancels due to the anti-symmetry of the Riemann tensor in the first two or last two indices.

The boundary area of the sphere is then

$$A_{curved} = \int_{\text{AllAngles}} dA_{flat} \left(1 - \frac{\tau^2}{6} R_{\alpha\beta} t^\alpha t^\beta\right) = A_{flat} - \frac{\tau^2}{6} R_{\alpha\beta} \int_{\text{AllAngles}} dA_{flat} t^\alpha t^\beta. \quad (28)$$

We now need to integrate the term $t^\mu t^\nu$ over all angles. Clearly, since t^μ is a radial unit vector, once integrated over all angles the term should be rotationally invariant. Thus,

$$\int_{\text{AllAngles}} t^\mu t^\nu dA_{flat} = a g^{\mu\nu}$$

The proportionality constant can be found by contracting both sides and integrating,

$$A_{flat} = a D$$

so that

$$a = \frac{A_{flat}}{D}.$$

Equation 28 now simplifies to

$$A_{curved} = A_{flat} \left(1 - \frac{\tau^2}{6D} R\right).$$

We can then invert this to find

$$R = \lim_{\tau \rightarrow 0} \frac{6D}{\tau^2} \left(1 - \frac{A_{curved}}{A_{flat}}\right).$$

Which completes the proof of equation 8.

References

- [1] M. Nakahara, Geometry, Topology, and Physics, (Institute of Physics Publishing, 1996), Chapter 7.
- [2] C. Misner, K. Thorne, J. Wheeler, Gravitation, (W.H. Freeman and Company, 1973), Box 14.1, pg. 335-340.
- [3] R.M. Wald, General Relativity, (The University of Chicago Press, 1984).
- [4] John Baez, General Relativity Tutorial,
<http://math.ucr.edu/home/baez/gr/gr.html>, specifically The Ricci and Weyl Tensors, <http://math.ucr.edu/home/baez/gr/ricci.weyl.html>
- [5] R.P. Feynman, Feynman Lectures on Gravity,(Addison-Wesley Publishing Company, 1995), Section 11.2 pg. 153-154.